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1983 J. Phys. A: Math. Gen. 16 L235

(http://iopscience.iop.org/0305-4470/16/7/007)

J. Phys. A: Math. Gen. 16 (1983) L235–L237. Printed in Great Britain

LETTER TO THE EDITOR

The monopole's spinor fields

G A Ringwood

34 Melford Rd, London E11 4PS, UK

Received 17 January 1983

Abstract. It is shown that the Wu–Yang monopole is described, completely geometrically, by the Kustaanheimo–Stiefel bundle. Sections of the tangent space to the bundle are spinor fields, showing that this monopole is an intrinsically spin one half particle.

In order to regularise the Coulomb problem Kustaanheimo and Stiefel (1965) construct a map from R^4 into physical space R^3. (For brevity, Kustaanheimo-Stiefel map will be abbreviated KS map.) The map is given by $x^i = S^i{}_\alpha q^\alpha$ (lower case Latin indices run from 1 to 3 and Greek indices run from 1 to 4) where S is the matrix

$$\begin{pmatrix} q^3 & q^4 & q^1 & q^2 \\ -q^2 & -q^1 & q^4 & q^3 \\ -q^1 & q^2 & q^3 & -q^4 \end{pmatrix}.$$

The nature of this map is clarified by using polar coordinates for R^3

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = r \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\,\sin\phi \\ \cos\theta \end{pmatrix}$$

and almost Cayley–Klein parameters for R^4

$$\begin{pmatrix} q^1 \\ q^3 \\ q^3 \\ q^4 \end{pmatrix} = r^{1/2} \begin{pmatrix} \sin\frac{1}{2}\theta\ \sin(\alpha-\frac{1}{2}\phi) \\ \cos\frac{1}{2}\theta\ \cos(\alpha+\frac{1}{2}\phi) \\ \cos\frac{1}{2}\theta\ \sin(\alpha+\frac{1}{2}\phi) \\ \sin\frac{1}{2}\theta\ \cos(\alpha-\frac{1}{2}\phi) \end{pmatrix}$$

where α runs from 0 to 2π. Restricted to $r=1$, this is a map from the unit sphere S^3 in R^4 to the unit sphere S^2 in R^3. For fixed θ and ϕ, variation of α describes a great circle on S^3. All points on such great circles are mapped into a single point of S^2. This restriction is the Hopf fibration: $S^3/S^1 = S^2$. (Note that a relation between the Hopf fibration and the Wu–Yang monopole has already been established (Ryder 1980).)

That the above describes a U1 principal bundle can be seen as follows. A point of R^4 can be coordinated by a pair of complex numbers

$$z = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = \begin{pmatrix} q^1 + \mathrm{i}q^4 \\ q^3 + \mathrm{i}q^2 \end{pmatrix} \in C^2.$$

The KS map κ is given by $x^i = \bar{z}\sigma^i z$, where the σ^i are the Pauli spin matrices. The free action of U1 on $R^4\backslash\{0\}$ (R^4 with the origin amputated) can be defined by $z \to uz$

where u is an element of U1. In terms of almost Cayley–Klein parameters

$$z = \mathrm{i}r^{1/2}\begin{pmatrix}\sin\frac{1}{2}\theta\exp[-\mathrm{i}(\alpha-\frac{1}{2}\phi)]\\ \cos\frac{1}{2}\theta\exp[-\mathrm{i}(\alpha+\frac{1}{2}\phi)]\end{pmatrix}$$

so that the orbits of U1 are the points of R^4 identified under κ. (Note that the orbit of U1 is the opposite orientation of increasing α.) The KS map realises $R^3\backslash\{0\}$ as a quotient of $R^4\backslash\{0\}$ by the free action of a one-parameter subgroup $\mathrm{U1}\simeq\mathrm{SO2}$ of the full rotation group SO4 of $R^4\backslash\{0\}$. Thus the KS map is the projection map of a principal U1 bundle with total space $R^4\backslash\{0\}$ and base $R^3\backslash\{0\}$. The Wu–Yang monopole can be described by a U1 bundle over $R^3\backslash\{0\}$. This letter will show that it is the KS bundle. Furthermore, it will be seen that the Wu–Yang monopole can be described completely geometrically and that it is naturally associated with a spinor bundle. The subsets $U_\mathrm{a}=\{R^3\backslash x^1=x^2=0, x^3\leqslant 0\}$ (R^3 with the closed negative half of the x^3 axis amputated) and $U_\mathrm{b}=\{R^3\backslash x^1=x^2=0, x^3\geqslant 0\}$ will provide a convenient open cover for the trivialisation of the KS bundle. The corresponding local sections are

$$s_\mathrm{a}: U_\mathrm{a}\to R^4\backslash\{0\}: r,\theta,\phi\to r^{1/2}\begin{pmatrix}\sin\frac{1}{2}\theta\\ \cos\frac{1}{2}\theta\exp -\mathrm{i}\phi\end{pmatrix}$$

$$s_\mathrm{b}: U_\mathrm{b}\to R^4\backslash\{0\}: r,\theta,\phi\to r^{1/2}\begin{pmatrix}\sin\frac{1}{2}\theta\exp\mathrm{i}\phi\\ \cos\frac{1}{2}\theta\end{pmatrix}.$$

The action of SU2 on $R^4\backslash\{0\}$ can also be defined by $z\to uz$ where $u\in\mathrm{SU2}$. Clearly the actions of SU2 and U1 commute. The SU2 action on $R^4\backslash\{0\}$ induces the action of SO3 on $R^3\backslash\{0\}$, as can be seen as follows. The action of SO3 on $R^3\backslash\{0\}$ can be described by $x^i\sigma^i\to ux^i\sigma^i u^{-1}$ for $u\in\mathrm{SU2}$. There is an identity

$$(\bar{z}\sigma^i z)\sigma^i = 2z\cdot\bar{z}+(\bar{z}z)I$$

where I is the unit 2×2 matrix and $z\cdot\bar{z}$ is the 2×2 matrix with components $z^A\bar{z}^B$ (upper case Latin indices take on the values 1 and 2). Using this identity the SO3 transformation becomes

$$x^i\sigma^i\to ux^i\sigma^i u^{-1} = 2uz\cdot\bar{z}u^{-1}-(\bar{z}z)I=(\bar{z}u^{-1}\sigma^i uz)\sigma^i$$

proving the claim.

The one-form $(\bar{z}z)^{-1}\bar{z}\,\mathrm{d}z$ is naturally invariant under the action of U1 and SU2:

$$((\overline{uz})uz)^{-1}\overline{uz}\,\mathrm{d}(uz)=(\bar{z}z)^{-1}\bar{z}\,\mathrm{d}z=(2r)^{-1}\,\mathrm{d}r-\mathrm{i}(\mathrm{d}\alpha+\tfrac{1}{2}\cos\theta\,\mathrm{d}\phi).$$

A form is said to be horizontal if it annihilates the kernel of $\mathrm{D}\kappa$, the derivative of κ. The kernel of $\mathrm{D}\kappa$, the space of vertical vectors, is spanned by ∂_α, the tangent vector to the great circles formed by the variation of α. The horizontal forms are spanned by $\mathrm{d}x^i=2S^i{}_\beta\mathrm{d}q^\beta$. The form $(2r)^{-1}\,\mathrm{d}r$ is horizontal but the form $\omega=-\mathrm{i}(\mathrm{d}\alpha+\frac{1}{2}\cos\theta\,\mathrm{d}\phi)$ is not: $\langle\omega,\partial_\alpha\rangle=-\mathrm{i}$. This latter form constitutes a connection on the principal bundle. On local sections $s_\mathrm{a}(\alpha=\frac{1}{2}\phi)$ and $s_\mathrm{b}(\alpha=-\frac{1}{2}\phi)$, ω becomes $-\frac{1}{2}(\cos\theta+1)\,\mathrm{d}\phi$ and $-\frac{1}{2}(\cos\theta-1)\,\mathrm{d}\phi$ respectively. In physical interpretation the bundle $R^4\backslash\{0\}$ is the space of phase factors, the base $R^3\backslash\{0\}$ is physical three-space and U1 is the gauge group. The form ω is a gauge potential which describes a Wu–Yang monopole of charge 1.

Defining complex tangent vectors $\partial_{\bar{z}}=\frac{1}{2}(\partial_{q^1}+\mathrm{i}\partial_{q^4},\partial_{q^3}+\mathrm{i}\partial_{q^2})$, the derivative $\mathrm{D}\kappa$, the map from the tangent space over a point in $R^4\backslash\{0\}$ to the tangent space over the corresponding point in $R^3\backslash\{0\}$, is given by $\partial_z\psi=\partial_{z^A}\psi^A\to\bar{z}\sigma^i\psi\partial_{x^i}$. Consider a rotation of the coordinates in $R^3\backslash\{0\}$ generated by $u\in\mathrm{SU2}$: $x^i\sigma^i=ux'^i\sigma^i u^{-1}$. Then, $\bar{z}\sigma^i\psi\partial_{x^i}=\overline{uz}\sigma^i u\psi\partial_{x'^i}$ so that $z'=uz$ and $\psi'=u\psi$. Thus the components of the vector fields over $R^4\backslash\{0\}$ with respect to ∂_z transform in the way one expects spinors over $R^3\backslash\{0\}$ to transform under a rotation of the coordinates. Thus sections of a tangent vector field over $R^4\backslash\{0\}$ behave as spinors over $R^3\backslash\{0\}$.

There is a natural metric (Ringwood and Devreese 1980) on $R^4\backslash\{0\}$, invariant under U1 and SU2, which extends the pullback of the natural metric on $R^3\backslash\{0\}$,

$$4(\bar{z}z)\,\mathrm{d}\bar{z}\,\mathrm{d}z = \mathrm{d}x^i\,\mathrm{d}x^i + 4r^2\bar{\omega}\omega = \mathrm{d}r\,\mathrm{d}r + r^2(\mathrm{d}\theta\,\mathrm{d}\theta + \sin^2\theta\,\mathrm{d}\phi\,\mathrm{d}\phi + 4\bar{\omega}\omega).$$

The term in brackets is the natural metric on S^3. (Note that in spherical coordinates this appears to be the natural flat metric on R^4, but this is not the case as the radial variable is $r^{1/2}$ and not r.) This metric is not Kähler but it is conformally flat:

$$4(\bar{z}z)\,\mathrm{d}\bar{z}\,\mathrm{d}z = 4q^\beta q^\beta\,\mathrm{d}q^\alpha\,\mathrm{d}q^\alpha.$$

With this metric the space orthogonal to the vertical vectors is the (dual) complement of ω, the space of horizontal vectors, so the metric gives the natural splitting of the tangent space.

It is clear from above that the Wu–Yang monopole can be completely described in geometrical terms with no physical assumptions. This interpretation is reminiscent of Kaluza–Klein unification. The tangent space of the KS bundle is a geometrical spinor bundle for the monopole. It has been previously shown that if one is prepared to regard some aspects of the gauge field as observable, e.g. charge (global gauge transformations), then it is possible to define topological concepts of spin (Ringwood and Woodward 1981) and statistics (Ringwood and Woodward 1982) for the 't Hooft–Polyakov monopoles. It is notable that the concept of spin associated with topological particles, kinks (Williams 1970), 't Hooft–Polyakov monopoles and now Wu–Yang monopoles is essentially due to the Hopf map. The present association of spin with the Wu–Yang monopole is, however, rather different from the previous two. Firstly, it is geometrical (not just topological), and secondly, while the previous associations showed that the topological particles could admit half-odd-integral spin, the integer unspecified, the present association is one half spin. It has been shown above how the Wu–Yang monopole describes a topological charged particle with intrinsic spin one half. Because of the duality between electric and magnetic fields, the Wu–Yang monopole could describe an electric monopole equally as well as a magnetic monopole. In the former case the quantisation of electric charge is a direct geometric consequence and it is unnecessary to invoke quantum mechanics to find an explanation for electric charge quantisation and spin.

The author wishes to express his gratitude to J T Devreese and the Department of Physics, University of Antwerp UIA, Belgium where part of this work was carried out.

Note added in proof. The metric $\mathrm{d}x^i\,\mathrm{d}x^i + \bar{\omega}\omega$ is more appropriate as its geodesics project to the trajectories of charged test particles on R^3.

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